SOGANG-HEP-206/96

Entropy of the BTZ Black Hole in 2+1 Dimensions

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(April 1996, revised version)

Abstract

We semi-classically calculate the entropy of a scalar field in the background of the BTZ black hole, and derive the perimeter law of the entropy. The proper length from the horizon to the ultraviolet cutoff is independent of both the mass and the angular momentum of the black hole. It is shown that the superradiant scattering modes give the sub-leading order contribution to the entropy while the non-superradiant modes give the leading order one, and thus superradiant effect is minor.

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Two decades ago, Bekenstein suggested that the entropy of a black hole is proportional to the area of the horizon through the thermodynamic analogy [1]. Subsequently, Hawking showed that the entropy of the Schwarzschild black hole satisfies exactly the area law by means of Hawking radiation based on the quantum field theory [2]. On the other hand, 't Hooft has argued that when one calculates the black hole entropy, the modes of a quantum field in the vicinity of a black hole horizon should be cut off due to gravitational effects rather than infinitely piling up by imposing a brick-wall cutoff just beyond the horizon. Nowadays the evaluation of black hole entropy in terms of statistical mechanics is one of the outstanding quantum gravity problems in connection with the information loss problem [3,4].

On the other hand, there has been much interest in lower dimensional theories of gravitation with the aim of the consistent quantum gravity. In a (1+1)-dimensional gravity, Callan-Giddings-Harvey-Strominger (CGHS) model [5], which has been improved by Russo, Susskind, and Thorlacius (RST), gives the analytic solution of an evaporating black hole in the semi-classical approximation [6]. Recent studies on the black hole thermodynamics based on these models [7–9] show that the entropy satisfies the area law and thermodynamic second law. For the (2+1)-dimensional anti-de Sitter gravity, Bañados, Teitelboim, and Zanelli (BTZ) have obtained the black hole solution, which is asymptotically anti-de Sitter rather than asymptotically flat and is characterized by the mass and the angular momentum [10]. For the consistent formation of the event horizon, the angular momentum should be restricted to some values. The various thermodynamic properties of the BTZ black hole were shown in Refs. [11,12]. Similar to the (1+1)-dimensional dilation gravity [13], there is no dynamically propagating degrees of freedom in contrast to the four dimensional Einstein gravity. Thus the BTZ black hole can also be a good candidate in studying the quantum aspects of black holes without the complexity of degrees of freedom.

Once the BTZ black hole is assumed, it is natural to consider some particles or fields around the black hole due to the quantum fluctuation of particles so called Hawking radiation. Therefore one can consider the black hole-matter coupled action as a total system.

It would be interesting to confirm the area law for the BTZ black hole since it can be an universal property of black holes. As well-known, the gravitational part of the entropy [14] was already calculated in [10,15,16], and the area law is satisfied. However, we could not assert generally the area law of black holes, such that in Ref. [17] the entropy of matters on the BTZ black hole does not seem to satisfy the perimeter law. Moreover, the geometrical structure of the BTZ black hole in 2+1 dimensions is somewhat different from the usual Schwarzschild black hole (or Kerr black hole) in four dimensions.

In this paper, we shall recast the entropy of a matter on the BTZ black hole background. By using the brick wall method developed by 't Hooft [18], we shall consider a Klein-Gordon field on the BTZ black hole, and obtain the free energy in the semi-classical approximation. We shall consider the entropy for the non-rotating and the rotating case separately. For the non-rotating black hole, we obtain the free energy which gives the desired entropy formula. For the rotating case, the free energy is composed of two pieces, one is the non-superradiant (NSR) part and the other is the superradiant (SR) part. It is shown that the NSR part gives the leading order contribution to the free energy in terms of the brick wall cutoff while the SR part gives the sub-leading order one. It is also shown that, although the SR part contains a divergent term due to the large angular quantum number, which is absent in the NSR part, it does not contribute to the entropy. After all, for both cases, we obtain the entropy expressed in terms of the perimeter by choosing the intrinsic ultraviolet cutoff, which is independent of mass and angular momentum of the black hole. This result is similar to the four dimensional case obtained by 't Hooft.

Let us now introduce the (2+1)-dimensional gravity, which is given by

$$I = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left[R + \frac{2}{l^2} \right] + B \ , \tag{1}$$

where $\Lambda = -\frac{1}{l^2}$ is the cosmological constant and B is the boundary term. Then the classical equation of motion yields the BTZ metric as

$$ds^2 = g_{tt} dt^2 - J dt d\theta + \frac{1}{N^2(r)} dr^2 + r^2 d\theta^2, \tag{2}$$

$$g_{tt} = -\left(\frac{r^2}{l^2} - M\right), \tag{3}$$
$$N^2(r) = \frac{r^2}{l^2} - M + \frac{J^2}{4r^2}.$$

There exist two coordinate singularities corresponding to the outer and inner horizon from $N^2(r) = 0$,

$$r_\pm = \sqrt{M}l\left[\frac{1}{2}\left(1 \pm \sqrt{1 - \left(\frac{J}{Ml}\right)^2}\right)\right]^{1/2} \quad (\, |J| \leq Ml \,). \tag{4}$$

¿From now on, we shall consider only the nonextremal case, $r_+ \neq r_-$. We regard the black hole horizon as $r_H = r_+$, which has the non-rotating limit. In later convenient use, some quantities are rewritten by $r_\pm$,

$$N^2(r) = \frac{1}{l^2 r^2}(r^2 - r_+^2)(r^2 - r_-^2), \tag{5}$$
$$M = \frac{r_+^2 + r_-^2}{l^2}, \tag{6}$$
$$J = \frac{2r_+ r_-}{l}. \tag{7}$$

The stationary limit $r_{\rm erg}$ which is defined by the radius of the ergosphere, is obtained by solving $g_{tt} = 0$ as follows

$$r_{\rm erg} = \sqrt{M}l = \sqrt{r_+^2 + r_-^2}. \tag{8}$$

The angular velocity of the black hole horizon is defined by

$$\Omega_H = -\left.\frac{g_{t\theta}}{g_{\theta\theta}}\right|_{r=r_+} = \frac{J}{2r_+^2} = \frac{r_-}{lr_+}. \tag{9}$$

Let us now introduce a Klein-Gordon field equation on the BTZ black hole background,

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) - \mu^2\Phi = 0, \tag{10}$$

where μ is the mass of a scalar field Φ. The above equation can be solved through the separation of variables, i.e., we can write the wave function as

$$\Phi(r, \phi, t) = e^{-iEt}e^{im\phi}R_{Em}(r), \tag{11}$$

where m is the azimuthal quantum number. Then, the radial equation becomes

$$\frac{1}{r}\partial_r[rN^2(r)\partial_r R_{Em}(r)] + N^2(r)k^2(r,m,E)R_{Em}(r) = 0 \ , \tag{12}$$

where the r-dependent radial wave number is given by

$$k^2(r,m,E) = \frac{1}{N^4(r)}\left[E^2 - \mu^2 N^2(r) - \frac{JEm}{r^2} + \frac{m^2(M - r^2/l^2)}{r^2}\right] \tag{13}$$

in the WKB approximation [18,19]. According to the semi-classical quantization rule, the radial wave number is quantized as

$$\pi n_r = \int_{r_H+\epsilon}^{L} dr k(r,m,E) \ , \tag{14}$$

where n_r is assumed to be a nonnegative integer, and ϵ and L are ultraviolet and infrared regulators, respectively. This is nothing but the quantization condition of energy since $E = E(n_r, m)$ by inverting the relation (14). The number of modes with energy not exceeding E is obtained by assuming the first brick wall (ultraviolet regulator) to be located at just outside of the outer horizon. Accordingly, we consider the radial integration for $r > r_{\rm H}$. On the other hand, for sufficiently large r the radial wave vector k can be a complex value unless we put the infrared regulator as $r_{\rm max} = L \approx \frac{1}{\mu}$. In later calculation for the free energy, we shall set $L \to \infty$ for the massless scalar particle $\mu^2 = 0$ for simplicity without losing consistency.

The free energy at inverse temperature β on the rotating black hole with the angular velocity Ω_H is represented by [17,20]

$$e^{-\beta F} = \prod_K \left[1 - e^{-\beta(E_K - m\Omega_H)}\right]^{-1} \ , \tag{15}$$

where K represents the set of quantum numbers. Note that the dependence on $E_K - m\Omega_H$ rather than E_K is a sign that the BTZ black hole has superradiant scattering modes for the rotating case. By using Eq. (14), the free energy can be rewritten as

$$\begin{aligned} F &= \frac{1}{\beta}\sum_K \ln\left[1 - e^{-\beta(E_K - m\Omega_H)}\right] \ \approx \ \frac{1}{\beta}\int dn_r \int dm \ \ln\left[1 - e^{-\beta(E - m\Omega_H)}\right] \\ &= -\int dm \int dE \ \frac{n_r}{e^{\beta(E - m\Omega_H)} - 1} \\ &= -\frac{1}{\pi}\int dm \int dE \frac{1}{e^{\beta(E - m\Omega_H)} - 1}\int_{r_H+\epsilon}^{L} dr k(r,m,E) \ , \end{aligned} \tag{16}$$

where we have taken the continuum limit in the first line and integrated by parts in the second line in Eq. (16).

We first consider the non-rotating black hole. Since it does not rotate, $\Omega_H = 0$, $J = 0$, or $r_- = 0$ as can be seen from Eq. (9). The free energy (16) then becomes

$$F_{(J=0)} = -\frac{1}{\pi} \int dm \int_0^\infty dE \frac{1}{e^{\beta E} - 1} \int_{r_H+\epsilon}^L dr k(r, m, E) \ . \tag{17}$$

For the evaluation of the physical free energy, the guideline provided by the brick wall method is to make the result of integration be real. Following this line, if we perform m-integration of Eq. (17) firstly and the remaining radial and energy integrations, then the form of the free energy is simply obtained as follows

$$\begin{aligned} F_{(J=0)} &= -\frac{1}{2} \int_0^\infty dE \frac{1}{e^{\beta E} - 1} \int_{r_H+\epsilon}^L dr \frac{lr}{(r^2 - r_+^2)^{3/2}} \left[l^2 E^2 - \mu^2 (r^2 - r_+^2) \right] \\ &= \frac{l\mu^2}{2\beta} \int_0^\infty dz \frac{1}{e^z - 1} (L - \sqrt{(r_H + \epsilon)^2 - r_+^2}) \\ &\quad + \frac{\zeta(3) l^3}{\beta^3} \left(\frac{1}{L} - \frac{1}{\sqrt{(r_H + \epsilon)^2 - r_+^2}} \right) , \end{aligned} \tag{18}$$

where $z \equiv \beta E$. Now, if the scalar field is massless, i.e., $\mu = 0$, for simplicity, and the limit $L \to \infty$ is taken, the free energy (18) is simplified as follows

$$F_{(J=0)} = -\frac{\zeta(3) l^3}{\beta^3} \frac{1}{\sqrt{(r_H + \epsilon)^2 - r_+^2}} , \tag{19}$$

which is exact in the sense of WKB approximation.

Let us now turn to the evaluation of the entropy for the massless field, which can be obtained from the free energy (19) at the black hole temperature. Since we are considering the non-rotating black hole and will also deal with the rotating one later, we designate the entropy for the non-rotating case as $S_{(J=0)}$, while for the rotating case as $S_{(J\neq 0)}$. For the non-rotating case, the inverse of Hawking temperature, β_H, is given by

$$\beta_H = \frac{2\pi l^2}{r_+}, \tag{20}$$

then the entropy is

$$
\begin{aligned}
S_{(J=0)} &= \beta^2 \frac{\partial F_{(J=0)}}{\partial \beta}\bigg|_{\beta=\beta_H} \\
&= \frac{3\zeta(3) l^3}{\beta_H^2} \frac{1}{\sqrt{(r_H+\epsilon)^2 - r_+^2}} \quad .
\end{aligned} \tag{21}
$$

This result clearly shows that the entropy behaves like $1/\sqrt{\epsilon}$ as $\epsilon \to 0$, that is, it is divergent in terms of the ultraviolet brick wall cutoff ϵ. At this level, ϵ is not determined. However, by requiring that the entropy $S_{(J=0)}$ satisfies a statement about entropy, i.e. the area law, the brick wall cutoff ϵ may be determined as a finite value. The process of determining ϵ may be reversed: choosing ϵ making $S_{(J=0)}$ satisfy the area law. If we now choose the cutoff as

$$
\epsilon = r_+ \left(\sqrt{1 + \frac{a^2}{l^2}} - 1 \right) \ , \tag{22}
$$

where the constant a is defined by

$$
a \equiv \frac{3\zeta(3)}{2^4 \pi^3} \tag{23}
$$

and its numerical value is approximately $a \approx 7.3 \times 10^{-3}$, then the entropy (21) satisfies the area (perimeter) law,

$$
S = 2 \cdot 2\pi r_+ . \tag{24}
$$

The cutoff (22) seems to depend on the mass of the black hole. But the invariant cutoff will be independent of it as far as we require the perimeter law of the entropy. The invariant distance from the horizon to the brick wall is calculated by definition as

$$
\begin{aligned}
\tilde{\epsilon} &= \int_{r_+}^{r_H+\epsilon} \frac{1}{N(r)} dr \\
&= l \ln \left(\frac{\sqrt{(r_H+\epsilon)^2 - r_+^2} + (r_H+\epsilon)}{r_+} \right) .
\end{aligned} \tag{25}
$$

After some calculations, the entropy (21) is neatly represented in terms of the invariant cutoff (25) as follows,

$$
S_{(J=0)} = \frac{4\pi a}{l} \frac{r_+}{\sinh\left(\frac{\tilde{\epsilon}}{l}\right)} . \tag{26}
$$

This entropy also satisfies the perimeter law if we identify the invariant cutoff as

$$\tilde{\epsilon} = l \sinh^{-1}\left(\frac{a}{l}\right) = l \ln\left(\frac{a}{l} + \sqrt{1 + \left(\frac{a}{l}\right)^2}\ \right), \tag{27}$$

where as expected $\tilde{\epsilon}$ is just a constant, and independent of the mass and angular momentum of the black hole. This feature appears to be an intrinsic property of horizon [18]. Consequently, the entropy and invariant cutoff are all finite.

We now turn to the case of rotating black hole and deal with it following the same steps adopted in the non-rotating case. The scalar field is taken to be massless, $\mu = 0$. Since the black hole has now angular velocity, the modes of scalar field are divided into two kinds of modes, which are superradiant (SR) and non-superradiant (NSR) modes. The SR modes are the common feature of rotating black holes like a Kerr black hole, and are characterized by modes of $0 \leq E \leq m\Omega_H$ and $m > 0$ [20], while the NSR modes are those of $E > m\Omega_H$ and any m. These two kinds of modes are distinct. Thus, we should divide the range of energy integration in the expression of free energy (16) as

$$\int dE = \int_{m\Omega_H}^{\infty} dE + \int_0^{m\Omega_H} dE \tag{28}$$

Then the free energy is divided into two parts: non-superradiant part ($F_{(NSR)}$) and superradiant part ($F_{(SR)}$),

$$F_{(J\neq 0)} = F_{(NSR)} + F_{(SR)}, \tag{29}$$

where we designate the free energy for the present case as $F_{(J\neq 0)}$ to distinguish it from that of the non-rotating case.

The NSR part of the free energy is given by

$$\begin{aligned} F_{(NSR)} &= -\frac{1}{\pi}\int dm \int_{m\Omega_H}^{\infty} dE \frac{1}{e^{\beta(E-m\Omega_H)}-1}\int_{r_H+\epsilon}^{L} dr k(r,m,E) \\ &= -\frac{1}{\pi}\int dm \int_0^{\infty} dE \frac{1}{e^{\beta E}-1}\int_{r_H+\epsilon}^{L} dr k(r,m,E+m\Omega_H)\ , \end{aligned} \tag{30}$$

where at the second line changing of variable, $E \to E + m\Omega_H$, was performed in order to make the integrations for m and E become independent. This expression for the NSR part

is of the same form as that of Eq. (17) except the details of the radial wave number k. The integrations in Eq. (30) can be evaluated straightforwardly and exactly. Similar to the non-rotating case, the guideline for the evaluation of the physical free energy is to make the result of integration be real. Then the free energy is obtained as follows

$$\begin{aligned} F_{(NSR)} &= -\frac{1}{2}\int_0^\infty dE \frac{1}{e^{\beta E}-1}\int_{r_H+\epsilon}^L dr \frac{lE^2}{r(r_+^2-r_-^2)N^2(r)}\frac{r_+^3(r^2-r_-^2)}{\sqrt{(r_+^2-r_-^2)(r^2-r_+^2)}} \\ &= \frac{\zeta(3)l^3}{\beta^3}\left(\frac{r_+^2}{r_+^2-r_-^2}\right)^{3/2}\left(\frac{1}{L}-\frac{1}{\sqrt{(r_H+\epsilon)^2-r_+^2}}\right) \\ &\overset{L\to\infty}{=} -\frac{\zeta(3)l^3}{\beta^3}\left(\frac{r_+^2}{r_+^2-r_-^2}\right)^{3/2}\frac{1}{\sqrt{(r_H+\epsilon)^2-r_+^2}}\ , \end{aligned} \tag{31}$$

where we have calculated the free energy for the nonextremal case so far.

On the other hand, the SR part of the free energy is given by

$$\begin{aligned} F_{(SR)} &= -\frac{1}{\pi}\int_{>0} dm \int_0^{m\Omega_H} dE \frac{1}{e^{\beta(E-m\Omega_H)}-1}\int_{r_H+\epsilon}^L dr k(r,m,E) \\ &= -\frac{\Omega_H}{\pi}\int_{>0} dm m^2 \int_0^1 dx \frac{1}{e^{-\beta m\Omega_H(1-x)}-1} \\ &\quad \times \int_{r_H+\epsilon}^L dr \frac{1}{lr^2N^2(r)}\left[\frac{r_-^2}{r_+^2}(r^2x-r_+^2)^2-(r^2-r_+^2)(r^2-r_-^2)\right]^{1/2}\ , \end{aligned} \tag{32}$$

where the change of variable, $E\to xm\Omega_H$, was performed at the second line. As in the previous evaluations, we first consider the m-integration. In the present case, unlike the previous ones, we encounter a new situation that each value of the azimuthal quantum number m ranging from zero to infinity may contributes to $F_{(SR)}$. This means that for any value of m, the SR part of the free energy $F_{(J\neq 0)}$ is real and thus physical, and there is no upper bound in the m-integration. However, as we can see from Eq. (32), the integration of the variable m leads to the divergent result. A certain factor, regulating factor, is thus needed to regulate the m-integration. Note that the m-integration appearing in the calculations of $F_{(J=0)}$ and $F_{(NSR)}$ had bounded ranges of integration variable and hence any regulating process was not needed. We now take the regulating factor as $e^{-\epsilon_m m}$ and consider this in the m-integration. The evaluation of integration is straightforward and the result is as follows.

$$\int_0^\infty dm \frac{m^2\, e^{-\epsilon_m m}}{e^{-\beta\Omega_H(1-x)m}-1} = -\frac{2}{\epsilon_m^3} - \frac{2\zeta(3)}{[\beta\Omega_H(1-x)]^3}\ , \tag{33}$$

which is obviously divergent as the regulating infinitesimal parameter ϵ_m goes to zero, $\epsilon_m \to 0$. Next, what we are going to do is the evaluation of the x-integration. The variable x ranges from zero to one. Due to the presence of the second term on the right hand side of Eq. (33), the integrand entered in the x-integration diverges at $x = 1$, and thus the x-integration in Eq. (32) becomes divergent. This divergence is related to that at $E = m\Omega_H$, since $x = 1$ corresponds to $E = m\Omega_H$. However, as we shall see later, this divergence does not appear in the final expression of $F_{(SR)}$, which is obtained by requiring that it is to be real. Since there is a divergence at $x = 1$, we introduce a new infinitesimal regulating cutoff ϵ_E in the x-integration as follows:

$$\int_0^{1-\epsilon_E} dx\ . \tag{34}$$

With this prescription, we may evaluate the x-integration and expand the result in powers of ϵ_E. The expansion shows that there are two divergent terms as follows:

$$\frac{1}{2\epsilon_E^2}\sqrt{C} - \frac{1}{\epsilon_E}\frac{r_-^2 r^2(r^2-r_+^2)}{r_+^2\sqrt{C}}\ , \tag{35}$$

where

$$C \equiv \frac{r^2}{r_+^2}(r_+^2 - r_-^2)(r_+^2 - r^2)\ . \tag{36}$$

The definition of C shows that C is less than zero for $r > r_+$. This means that $\sqrt{C}$ is always imaginary for the value of r ranging from $r_+ + \epsilon$ to L. Thus divergent terms in Eq. (35) do not contribute to the SR part of the physical free energy, and only the terms of zeroth order in ϵ_E enter in our consideration. Even among the terms of ϵ_E^0-order, the number of terms making $F_{(SR)}$ real is restricted. We now turn to the evaluation of the last r-integration. Unlike the case of $F_{(NSR)}$ in Eq. (31), it may not be performed neatly and compactly. However, since what is needed is the divergence structure in terms of the brick wall cutoff ϵ, we expand the r-integration in powers of ϵ and evaluate only the terms which diverge as $\epsilon \to 0$. Then the SR part of the physical free energy is obtained as follows

$$
\begin{aligned}
F_{(SR)} = & \frac{l\Omega_H}{\pi}\int_{r_H+\epsilon}^{r_{\rm erg}} dr \frac{1}{(r^2-r_+^2)(r^2-r_-^2)}\left\{ \frac{r_+^2\sqrt{r_{\rm erg}^2-r^2}}{\epsilon_m^3 r} - \frac{\zeta(3)}{(\beta\Omega_H)^3}\left[\frac{r_+^2 r\sqrt{r_{\rm erg}^2-r^2}}{r_+^2-r_-^2}\right.\right. \\
& \left.\left. + \frac{r_+ r_-^2 r(r^2-r_+^2)(r^2-r_-^2)}{[(r_+^2-r_-^2)(r^2-r_+^2)]^{3/2}}\tan^{-1}\left(\frac{r_+\sqrt{r^2-r_+^2}}{\sqrt{r_+^2-r_-^2}\sqrt{r_{\rm erg}^2-r^2}}\right)\right]\right\} \\
\approx & \frac{l\Omega_H}{\pi}\frac{r_-}{r_+^2-r_-^2}\left[\frac{1}{2\epsilon_m^3} - \frac{\zeta(3)}{(\beta\Omega_H)^3}\frac{r_+^2}{r_+^2-r_-^2}\right]\ln\left(\frac{r_+}{\epsilon}\right) + \mathcal{O}(\epsilon^0)\ . \qquad (37)
\end{aligned}
$$

The maximum value of r in the r-integration changed to $r_{\rm erg}$ due to the result of the x-integration. This implies that only the SR modes in the radial region between the brick wall and stationary limit, i.e., ergosphere, contribute to the SR part of the physical free energy. Comparing with the NSR part $F_{(NSR)}$ of the free energy in Eq. (31), $F_{(SR)}$ in (37) shows two different aspects. First is the appearance of the divergent structure, $1/\epsilon_m^3$, due to the large azimuthal quantum number, which is absent in $F_{(NSR)}$. Second is that the leading order divergence in terms of the brick wall cutoff ϵ is logarithmic ($\ln\epsilon$), while it is power like ($1/\sqrt{\epsilon}$) in the case of $F_{(NSR)}$. Thus it may be concluded that the SR part gives the sub-leading order contribution to the free energy $F_{(J\neq 0)}$, while the NSR part gives the leading order one.

Before considering the entropy for the case of rotating black hole, we would like to comment on the non-rotating limit of Eq. (37). From the expression of $F_{(SR)}$ in Eq. (32), it is obvious that $F_{(SR)}$ is zero when the limit $\Omega_H \to 0$ is taken. It is natural and implies that the SR modes do not exist in the non-rotating case, as it should be. However, $F_{(SR)}$ in Eq. (37) diverges under this limit. This inconsistency may be explained as follows: up to now, we have given a formulation retaining Ω_H as a non-zero finite value, or more precisely $r_- \neq 0$. That Ω_H is not zero means that the ergosphere does exist and hence $r_{\rm erg} > r_H$. If we now assume that Ω_H is so small such that $r_{\rm erg} < r_H + \epsilon$, it happens that we count the SR modes inside the brick wall, as can be known from the range of r-integration in Eq. (37). But, the essence of the brick wall method is to count the modes outside the brick wall. Therefore, in the framework of the method, the evaluation of $F_{(SR)}$ for the black hole rotating with angular velocity lower than a certain value becomes totally meaningless. The condition

which Ω_H has to satisfy may now be determined by looking at the inequality, $r_{\rm erg} > r_H + \epsilon$. After a short manipulation by using Eqs. (8) and (9), the condition is obtained as follows:

$$\Omega_H^2 > \frac{1}{l^2}\left[\left(1+\frac{\epsilon}{r_+}\right)^2 - 1\right] \ . \tag{38}$$

This may be also understood as the validity condition for the final expression of $F_{(SR)}$ in Eq. (37). The argument given up to now explains why we cope with the non-rotating and the rotating black holes individually.

We are now ready to evaluate the entropy for the massless field as for the case of the rotating black hole. The physical free energy $F_{(J\neq 0)}$ is obtained by substituting the results of Eqs. (31) and (37) into Eq. (29):

$$F_{(J\neq 0)} \approx -\frac{l^3\zeta(3)}{\sqrt{2}\beta^3}\left(\frac{r_+}{r_+^2 - r_-^2}\right)^{3/2}\frac{r_+}{\sqrt{\epsilon}} + \frac{l\Omega_H}{\pi}\frac{r_-}{r_+^2 - r_-^2}\left[\frac{1}{2\epsilon_m^3} - \frac{\zeta(3)}{(\beta\Omega_H)^3}\frac{r_+^2}{r_+^2 - r_-^2}\right]\ln\left(\frac{r_+}{\epsilon}\right) \ , \tag{39}$$

where the NSR part $F_{(NSR)}$ is expanded in powers of ϵ as $F_{(SR)}$ was. Unlike the free energy $F_{(J=0)}$ for the case of non-rotating black hole, $F_{(J\neq 0)}$ has an additional divergence due to the large azimuthal quantum number, which comes from the SR modes, besides that due to the brick wall cutoff. This additional divergence may be problematic when we consider the relation between the entropy and the brick wall cutoff. Fortunately, however, it does not matter because the entropy is obtained by differentiating the free energy by the inverse temperature β, but the term related to the divergence $1/\epsilon_m^3$ in Eq. (39) does not depend on β. Therefore, the entropy is safe from the additional divergence and, from the standard formula, is obtained as follows:

$$\begin{aligned} S_{(J\neq 0)} &= \beta^2 \left.\frac{\partial F_{(J\neq 0)}}{\partial \beta}\right|_{\beta=\beta_H} \\ &\approx \frac{3\zeta(3)}{4\pi^2 l}\left[\sqrt{\frac{r_+(r_+^2 - r_-^2)}{2}}\frac{1}{\sqrt{\epsilon}} + \frac{r_+^2}{\pi r_-}\ln\left(\frac{r_+}{\epsilon}\right)\right] \ , \end{aligned} \tag{40}$$

where β_H is for the rotating black hole and is given by

$$\beta_H = \frac{2\pi r_+ l^2}{r_+^2 - r_-^2} \ . \tag{41}$$

We now take only the leading order term in terms of ϵ in the r.h.s. of Eq. (40), which is the contribution from the NSR modes. Obviously, there is a contribution from the SR modes, which is however sub-leading and hence may be neglected. We now determine the value of ϵ by equating $S_{(J\neq 0)}$ with the area (perimeter) law (24). Then it is given by

$$\epsilon \approx \frac{r_+}{2}\left(\frac{a}{l}\right)^2\left(1-\frac{r_-^2}{r_+^2}\right) , \tag{42}$$

where a is the number defined in Eq. (23). The invariant distance from the horizon to the brick wall is calculated by using Eq. (25) and is obtained as follows:

$$\tilde{\epsilon} = a + \mathcal{O}(a^2) , \tag{43}$$

which is independent on the mass and angular velocity of the rotating black hole, as was in the case of the non-rotating black hole. It should be noted here that if the invariant distance of Eq. (27) is expanded in powers of a then it is just the same with Eq. (43) at the leading order. The entropy may now be rewritten in terms of $\tilde{\epsilon}$ as

$$S_{(J\neq 0)} \approx 4\pi a \frac{r_+}{\tilde{\epsilon}} , \tag{44}$$

which satisfies the perimeter law if we take Eq. (43) as the expression for $\tilde{\epsilon}$. Since the entropy for the non-rotating case also satisfies the perimeter law, it may be concluded that the perimeter law is always satisfied regardless of whether or not the black hole is rotating, at least in the framework of the brick wall method.

Before concluding remark, we would like to compare our results with those of ref. [17] where the authors investigate the thermodynamics of scalar fields in the BTZ black hole backgrounds and conclude that generically the entropy does not satisfy the perimeter law. As in our formulation, there appears three regularization cutoffs in ref. [17]: an infinitesimal distance from the horizon (ϵ_H), cutoff for the occupation number of the particle (N_1) for each mode satisfying $E - m\Omega_H \leq 0$, and cutoff for the absolute value of quantum number (N_2). Comparison between these cutoffs and those of us shows that ϵ_H is related to the brick wall cutoff ϵ, N_1 corresponds to ϵ_E related to the divergence at $E = m\Omega_H$, and N_2 corresponds

to the large azimuthal quantum number which leads to the introduction of the infinitesimal parameter ϵ_m. These three cutoffs remain in the final expression for the entropy, which may give difficulty in deriving the perimeter law by adjusting the cutoffs. However, as for our case, only the brick wall cutoff ϵ appears in the entropy while the other two parameters disappear during the formulation by natural and physical reasons, and the perimeter law is obtained by adjusting ϵ.

In conclusion, we have studied the entropy of a spinless and massless scalar field around the non-extremal BTZ black hole, which is similar to the Kerr metric but not asymptotically flat. The area law has been obtained by adjusting the location of the brick wall, which is just outside the horizon for both the cases of non-rotating and rotating black hole. As for the entropy in the rotating case, there was the contribution from the superradiant modes, which was however sub-leading, while the non-superradiant modes gave the leading order contribution. Although there have appeared three types of regulating parameters during our formulation, all of them except the brick wall cutoff ϵ were absent at the final expression of the entropy. It has been shown that, for the consistency of the brick wall method, the angular velocity of the rotating black hole should be greater than a certain value represented by Eq. (38). This led us to consider the rotating and the non-rotating black holes separately.

ACKNOWLEDGMENTS

S.-W. Kim and W. T. Kim were partially supported by Korea Science and Engineering Foundation, 95-0702-04-01-3. H. Shin and Y.-J. Park were supported in part by Ministry of Education, Project No. BSRI-95-2414. W. T. Kim was supported in part by the Korea Science and Engineering Foundation through the Center for Theoretical Physics(1996).

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